On a relation between the cycle packing number and the cyclomatic number of a graph

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Abstract. Let $G = (V, E)$ be a graph. A cycle packing $Z = \{C_1, \ldots, C_l\}$ of $G$ is a collection of pairwise edge-disjoint cycles $C_i$ of $G$ ($i = 1, \ldots, l$).

This paper deals with $\nu(G)$, the maximum cardinality of a cycle packing $Z$. Bounds on $\nu(G)$ are given, if $G$ is Eulerian. Moreover this number is related to the cyclomatic number $\gamma(G)$, the maximum number of independent cycles in $G$. A characterization of such graphs is given for which $\gamma(G) - \nu(G) = 1$ holds.

Key words. cycle packing, Eulerian graphs, extremal problems in graph theory

1. Introduction

In the following we consider an arbitrary undirected finite graph $G = (V, E)$. Let $|V(G)| = n$ and $|E(G)| = m$ denote the numbers of vertices and edges, respectively. If no confusion is possible we write $|V|$ and $|E|$. Sometimes an edge $e = (u, v) \in E(G)$ will also be understood as the graph $(\{u, v\}, \{e\})$ throughout this paper. A simple graph is a graph having neither loops nor different edges $e$ and $\tilde{e}$ with the same distinct endvertices. A graph in which all vertices $v$ have even degree $d(v)$ is called Eulerian. A graph $G' = (V', E')$ is called a subgraph of $G$, if $V' \subseteq V$ and $E' \subseteq E$. In this case we write $G' \subseteq G$. Similarly, a graph $G' = (V', E')$ is called a supergraph of $G$, if $G$ is a subgraph of $G'$. A subgraph $G|_{V'} \subseteq G$ is called an induced subgraph if $V(G|_{V'}) = V'$ and $E(G|_{V'}) = \{ e \in E(G) \mid \text{both endpoints of edge } e \text{ belong to } V' \}$. For $V' \subseteq V$ we define $G \setminus V' = G|_{V \setminus V'}$. A subgraph $G|_{E'} \subseteq G$ is induced by $E'$ if $V(G|_{E'}) = \{ v \in V(G) \mid v \text{ is incident to an edge } e \in E' \}$ and $E(G|_{E'}) = E'$. Two subgraphs $G' = (V', E')$, $G'' = (V'', E'') \subseteq G$ are called edge-disjoint if $E' \cap E'' = \emptyset$. For $E' \subseteq E$ we define $G \setminus E' = (V, E \setminus E')$. The union of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. For $V' \subseteq V$ we define the contraction $G|_{V'}$ with respect to $V'$ to be the graph resulting from $G$ by deleting all vertices of $V'$ and the edges of $G|_{V'}$, adding a new vertex $v'$ and replacing all edges $(u, v)$ with $u \notin V'$, $v \in V'$ by an edge $(u, v')$ (multiple edges may occur). A bridge $e \in E$ is an edge of a connected graph $G$ such that $G \setminus \{e\}$ is not connected. Furthermore we will make use of so called blocks, whereas a subgraph $H$ of an arbitrary graph $G$ is called a block, if $H$ is a connected
graph without a cut-vertex and there is no other graph $\tilde{H}$ with $H \subset \tilde{H} \subseteq G$ with this property. A vertex $v \in V$ is called a cut-vertex of a connected graph $G$ if $G \setminus \{v\}$ is not connected. Let $v_{i_1}, e_1, v_{i_2}, e_2, \ldots, e_r, v_{i_r}$ be a finite sequence of vertices $v_{i_j}$ and pairwise distinct edges $e_j = (v_{i_j}, v_{i_{j+1}})$ of $G$. Then the graph $W$ with $V(W) = \{v_{i_1}, v_{i_2}, \ldots, v_{i_r}\}$ and $E(W) = \{e_1, e_2, \ldots, e_r\}$ is called a walk from $v_{i_1}$ to $v_{i_r}$. If $W$ is closed (i.e. $v_{i_1} = v_{i_r}$) and a subgraph of $G$ we call it a circuit. A path is a walk in which all vertices $v$ have degree $d(v) \leq 2$. A closed path will be called a cycle.

Let $G = (V, E)$ be disaggregated into $r$ components $G_i, i = 1, \ldots, r$ and let $F$ be a maximal spanning forest of $G$, i.e. the union of spanning trees $T_i, i = 1, \ldots, r$ of the single components. An elementary cycle with respect to $F$ is a cycle $C_e$ that arises by adding some edge $e \in E \setminus E(F)$ to $E(F)$. The set of all elementary cycles with respect to $F$ is then given by $\{C_e \mid e \in E \setminus E(F)\}$. The cardinality of this set is independent of the choice of $F$ and is called the cyclomatic number of $G$. It is denoted by $\gamma(G)$ and determined by $\gamma(G) = m - n + r$.

A cycle packing $Z = \{C_1, \ldots, C_l\}$ of $G$ is a collection of $l$ pairwise edge-disjoint cycles in $G$. $Z$ is called maximal if $G \setminus \bigcup_{i=1}^l C_i$ is acyclic. This paper deals with properties of the cycle packing number

$$\nu(G) = \max\{l \mid \{C_1, \ldots, C_l\} \text{ is a cycle packing of } G\}.$$ 

The cycle packing problem consists of finding a maximum cycle packing $Z^*$, i.e. a cycle packing of cardinality $\nu(G)$.

Although cycle packings relate to a very natural question in graph theory, there are few papers that have focused attention on $\nu(G)$ systematically (see [4], [29], [34], [35], [36]). The cycle packing number appeared also in connection with the maximum Eulerian cycle decomposition problem ([6]) and in [8].

For given $l > 0$, the monograph [5] presents results on the minimal difference $|E| - |V|$ in order to guarantee the existence of $l$ edge-disjoint cycles in $G = (V, E)$. The book [20] refers to procedures for the construction of a maximal cycle packing for Eulerian graphs (which is in fact a decomposition). For this case it also gives a short description of a procedure (due to [26]) to get the set $S(G)$ of all maximal cycle packings of $G$.

In the beginning of the 90s a conjecture of Holyer ([24]) was proved that the problem to check whether the edge set of a graph $G$ can be decomposed into complete subgraphs $K_r$ of order $r$ is NP-complete for $r \geq 3$ (see [15]). The special case $r = 3$ leads to the problem of determining whether a triangle decomposition of $G$ exists. This result induces that the cycle packing problem is also NP-hard. Even for Eulerian graphs $G$ with maximum degree 4 it was shown in [6] that the cycle packing problem is NP-hard. Recent research therefore focusses on algorithms for the cycle packing problem with guaranteed approximation ratio (see [8], [7], [25]).

There are some results on $\nu(G)$ for a few classes of graphs $G$: If $G$ is a complete graph $K_n$, then its cycle packing number is related to the theory of Steiner triple systems [9]. The existence of such a system for a set of $n$ elements corresponds to the possibility of decomposing $K_n$ into 3-cycles. Such a decomposition is possible if and only if $n = 6k + 1$ or $n = 6k + 3, k \in \mathbb{N}$. Hence $\nu(K_{6k+1}) = 6k^2 + k$ and $\nu(K_{6k+3}) = 6k^2 + 5k + 1$. Using the idea of [21] for a construction of a Steiner triple system one obtains $\nu(K_{6k+5}) = 6k^2 + 9k + 3$ (see [12]). The observations of [30] for a 3-cycle decomposition of $K_{12} \setminus I$ with $I$ being the edge set of a 1-factor of the graph can be generalized for all non-Eulerian complete
graphs. We get \( \nu(K_{6k}) = 6k^2 - 2k \), \( \nu(K_{6k+2}) = 6k^2 + 2k \) and \( \nu(K_{6k+4}) = 6k^2 + 6k + 1 \).

The investigation of \( \nu(K_n) \) reveals its relation to Alspach’s problem ([2]): If \( n \) is odd and the integers \( a_1, a_2, \ldots, a_l \) satisfy \( \sum_{i=1}^{l} a_i = \frac{n(n-1)}{2} \), then there is a decomposition \( Z = \{C_1, C_2, \ldots, C_l\} \) of \( K_n \) with \( C_i \) being an \( a_i \)-cycle? If \( n \) is even and given integers \( a_1, a_2, \ldots, a_l \) satisfying \( \sum_{i=1}^{l} a_i = \frac{n(n-2)}{2} \), is there a decomposition \( Z = \{C_1, C_2, \ldots, C_l\} \) of \( K_n \) with \( C_i \) being an \( a_i \)-cycle?

For the class of dense Eulerian graphs it was proved in [3] that if \( n = |V(G)| \) is sufficiently large and the minimum degree \( \delta(G) \) satisfies \( \delta(G) \geq (1 - \epsilon)n \) there is an edge-disjoint decomposition of \( G \) in \( a_i \)-circuits if \( \sum_{i=1}^{l} a_i = |E(G)| \). Unfortunately, this result is not constructive for the determination of \( \nu(G) \).

For the case that \( G \) is planar Eulerian in [7] a polynomial time algorithm is given to obtain \( \nu(G) \)

Some results concerning directed graphs can also be found. In [28] and [32] the maximum number of pairwise edge-disjoint directed circuits of \( G \) is considered and put into relation to the smallest size of a set of edges that meet all directed circuits of \( G \). In [1] the authors prove a lower bound on this number, if \( G \) is an Eulerian directed graph.

Contributions dealing with a maximum packing of a graph \( G \) into vertex-disjoint cycles, i.e. the maximum number of cycles that pairwise have no vertex in common can e.g. be found in [11], [14], [17], [22], [37], [38]. For a survey, the paper [16] collects results on degree conditions that guarantee the existence of such a partition with a specified number of cycles. It should be pointed out that this maximum number in general has no obvious relation to \( \nu(G) \), since in a cycle packing \( Z \) the cycles \( C_i \) are not assumed to be vertex-disjoint.

There are also several papers dealing with the minimum number \( cd(G) \) of cycles required in a cycle covering of \( G \) (e.g. see [31], [18]). These contributions mainly focus their investigations on the conjecture of Hajós (see [27]), that \( cd(G) \leq \lfloor \frac{n-1}{2} \rfloor \) if \( G \) is a simple Eulerian graph on \( n \) vertices. For graphs \( G \) in this class with maximum degree 4 it is proved (see [19]) that \( cd(G) \leq \frac{n-1}{2} \).

2. Some bounds for Eulerian graphs

Obviously, the cycle packing number \( \nu(G) \) is well defined and finite, but a related maximum packing \( Z^* \) may not be unique. Let us start with some observations on the packing number, that can easily be checked.

**Proposition 1.** Let \( G = (V, E) \) be a graph.

(i) If \( G' \) is a subgraph of \( G \), then \( \nu(G') \leq \nu(G) \).

(ii) If \( e \in E(G) \) and \( G' = G \setminus \{e\} \), then \( \nu(G) - 1 \leq \nu(G') \leq \nu(G) \).

(iii) If \( e \in E(G) \) is a bridge in \( G \) and \( G' = G \setminus \{e\} \), then \( \nu(G') = \nu(G) \).

(iv) If \( e \in E(G) \) is a loop and \( G' = G \setminus \{e\} \), then \( \nu(G') = \nu(G) - 1 \).

(v) Let \( e \) and \( \bar{e} \) be two distinct edges in \( G \) that have the same distinct endvertices, and let \( G' \) denote the graph obtained from \( G \) by elimination of the cycle \( (e, \bar{e}) \). Then \( \nu(G') = \nu(G) - 1 \).

(vi) Let \( G_i \) be the components of \( G \) (\( i = 1, \ldots, r \)), then \( \nu(G) = \sum_{i=1}^{r} \nu(G_i) \).

(vii) Let \( B_i \) be the blocks of \( G \) (\( i = 1, \ldots, k \)), then \( \nu(G) = \sum_{i=1}^{k} \nu(B_i) \).
(viii) Let \( G_1 = (V, E_1) \) and \( G_2 = (V, E_2) \) be simple graphs and \( G' \) a supergraph of \( G_1 \cup G_2 \) with \( |E(G')| = |E_1| + |E_2| \) such that for every edge \( \bar{e} = (u, v) \in E_1 \cap E_2 \) there are two distinct edges \( e, \bar{e} \) in \( E(G') \) connecting \( u \) and \( v \). Then the inequality
\[
\nu(G') \geq \nu(G_1) + \nu(G_2)
\]
holds. For \( E_3 = E_1 \cap E_2 \) we get the more general inequality
\[
\nu(G') \geq \nu(G_1 \setminus E_3) + \nu(G_2 \setminus E_3) + |E_3|.
\]
(ix) If \( G \) is a simple connected graph, containing \( k \) vertices of odd degree, then there is an Eulerian subgraph \( G_\circ \) of \( G \) such that
\[
\nu(G_\circ) = \nu(G).
\]
(x) If \( G \) is a simple connected graph, containing \( k \) vertices of odd degree, then there is an Eulerian supergraph \( G_\circ \) of \( G \) that satisfies
\[
\nu(G_\circ) = \nu(G) + \frac{k}{2}.
\]
Due to property (vii) it is sufficient to study 2-vertex-connected graphs. Property (ii) gives reason to call an edge \( e \in E \) \( \nu \)-critical if \( \nu(G \setminus \{e\}) < \nu(G) \). If every edge of \( G \) is \( \nu \)-critical the graph is called \( \nu \)-critical. Using (x) we get

**Proposition 2.** A graph is \( \nu \)-critical if and only if it is Eulerian.

For an Eulerian graph its edge-set can be completely decomposed into edge-disjoint cycles. It is therefore apparent in (ix) and (x) that \( \nu(G) \) is related to the cycle packing numbers of corresponding Eulerian sub- and supergraphs of \( G \). The number \( \nu(G) \) sometimes might also be related to the cycle packing numbers of two special minimum Eulerian supergraphs\(^1\):

Assume that \( G \) is a simple connected graph that contains \( k > 0 \) vertices of odd degree. If \( G_1^\circ \) and \( G_2^\circ \) are two minimum Eulerian supergraphs of \( G \), then
\[
\nu(G_1^\circ) - \frac{k}{2} \leq \nu(G) \leq \nu(G_2^\circ) - 1.
\]

Hence, if we might identify two different minimum Eulerian supergraphs \( G_1^\circ \) and \( G_2^\circ \) of \( G \) satisfying \( \nu(G_1^\circ) - \nu(G_2^\circ) = \frac{k}{2} - 1 \), then \( \nu(G) = \nu(G_1^\circ) - \frac{k}{2} = \nu(G_2^\circ) - 1 \).

The following figure 1 illustrates an example of a graph \( G \) in which two of its minimum Eulerian supergraphs \( G_1^\circ, G_2^\circ \) satisfy \( |\nu(G_1^\circ) - \nu(G_2^\circ)| = \frac{k}{2} - 1 \), while figure 2 shows a graph\(^2\), in which all of its minimum Eulerian supergraphs have the same cycle packing number, i.e. a suitable pair \( G_1^\circ, G_2^\circ \) does not exist.

Note, that in order to look for suitable \( G_1^\circ \) and \( G_2^\circ \) of \( G \) we might have to consider \( \frac{k!}{2(\frac{k}{2})!} \) different minimum Eulerian supergraphs (together with their cycle packing numbers), corresponding to all possible partitions of the odd degree vertices into pairs.

\(^1\) An Eulerian supergraph \( G_1^\circ \) of a graph \( G \) is called a minimum Eulerian supergraph if there is no Eulerian supergraph \( G_2^\circ \) of \( G \) with \( |E(G_1^\circ)| > |E(G_2^\circ)| \).

\(^2\) This example was given by H. Sachs.
Fig. 1. Graph $G$ with two minimum Eulerian supergraphs $G_1, G_2$ satisfying $|\nu(G_1) - \nu(G_2)| = \frac{k}{2} - 1$

Fig. 2. Graph $G$ and all possible minimum Eulerian supergraphs $G_1, G_2, G_3$

A determination of $\nu(G)$ by properties (ix) or (x) or by finding suitable $G_1, G_2$ is therefore in general not successful.

Moreover, by a result of Caprara et al. [6] we get that the determination of the cycle packing number of an Eulerian graph is NP-hard, even if all its vertices have degree at most 4. In the special case of a plane Eulerian graph $G$ it was shown in [7] that there is a procedure of order $O(n \cdot m)$ for determining $Z^*$.

Using a result of Alon et. al. for directed graphs we can give a general lower bound for $\nu(G)$.

**Proposition 3.** Let $G = (V, E)$ be a simple connected Eulerian graph and $\delta = \min\{d(v) \mid v \in V\}$ its minimum degree. Then $\nu(G) \geq \frac{5}{4}\delta - 2$.

**Proof.** Let $T$ be an Eulerian tour in $G$. By orienting one edge of $T$ an orientation for $G$ is induced. Obviously, the minimum indegree $\delta^+$ and minimum outdegree $\delta^-$ in the oriented graph are equal to $\frac{1}{2}\delta$. [1] proved that a connected Eulerian directed graph with no parallel edges and with minimum outdegree $\delta^-$ contains a collection of $\frac{5}{2}\delta^- - 2 = \frac{5}{4}\delta - 2$ edge-disjoint cycles. This collection corresponds to a set of undirected edge-disjoint cycles in $G$ having the same cardinality.

For the class of simple connected Eulerian graphs bounds on $\nu(G)$ can be obtained by using the cyclomatic number $\gamma(G)$. Note, that the relation in Proposition 1, (vii) also holds for the cyclomatic number.

**Lemma 1.** Let $G$ be a plane Eulerian graph with components $G_i$, $i = 1, \ldots, r$. Then $\nu(G) \geq \frac{1}{2} (\gamma(G) + r)$.
Proof. It is well known that the $m_i - n_i + 2 = \gamma(G_i) + 1$ regions of each of the components $G_i$ (with $n_i$ vertices and $m_i$ edges) of a plane Eulerian graph $G$ are 2-colourable. Moreover, regions of the same colour in $G_i$ correspond to an independent set of vertices in the dual graph $\hat{G}_i$. Let $k_1^i$ and $k_2^i$ denote the particular numbers of the regions with the same colour, then $k_1^i + k_2^i = m_i - n_i + 2$. Hence $\nu(G_i) \geq \alpha(\hat{G}_i) \geq \max\{k_1^i, k_2^i\} \geq \frac{1}{2}(\gamma(G_i) + 1)$.

According to Proposition 1, (vi) we get

$$\nu(G) = \sum_{i=1}^{r} \nu(G_i) \geq \sum_{i=1}^{r} \frac{1}{2}(\gamma(G_i) + 1) = \frac{1}{2}\left(\sum_{i=1}^{r} \gamma(G_i) + r\right) = \frac{1}{2}(\gamma(G) + r),$$

where the last equality holds since $\gamma(G) = \sum_{i=1}^{r} \gamma(G_i)$.

Another bound on $\nu(G)$ can be obtained, when a maximal set of edge-disjoint circuits is considered. We get

**Proposition 4.** Let $G = (V, E)$ be a simple connected Eulerian graph and $\hat{Z} = \{\hat{C}_1, \ldots, \hat{C}_q\}$ a maximal set of edge-disjoint circuits in $G$.

(i) If $q \in \{1, 2\}$ and $\gamma(G) \geq 2q$ then $\nu(G) > q$.

(ii) If $G$ is planar, $q \geq 3$ and $\gamma(G) \geq 2q$ then $\nu(G) > q$.

**Proof.**

(i) Let $q = 1$. Since $2 \leq \gamma(G) = \frac{1}{2}\sum_{v \in V} d(v) - n + 1$, we have

$$\sum_{v \in V} \left(\frac{1}{2}d(v) - 1\right) \geq 1.$$

Therefore, there is at least one vertex $\bar{v} \in V$ such that $d(\bar{v}) \geq 4$. At the vertex $\bar{v}$ the circuit $\hat{C}_1$ can be decomposed into two edge-disjoint circuits $\hat{C}_1'$ and $\hat{C}_1''$ (with common vertex $\bar{v}$), i.e. $\nu(G) > 1$.

Now, let $q = 2$. Then $4 \leq \frac{1}{2}\sum_{v \in V} d(v) - n + 1$ and therefore

$$\sum_{v \in V} \left(\frac{1}{2}d(v) - 1\right) \geq 3.$$

If there is a vertex $\bar{v} \in V$ in one of the circuits, say $\hat{C}_1$, with $d|_{\hat{C}_1}(\bar{v}) \geq 4$, we can treat $\hat{C}_1$ according to the case $q = 1$ and enlarge the cardinality of $\hat{Z}$, i.e. $\nu(G) > 2$.

If no such vertex exists, we have $d|_{\hat{C}_k}(v) \leq 2$ for all $v \in V(\hat{C}_k)$ ($k = 1, 2$). But this implies, that there must be at least three distinct vertices $v_1, v_2, v_3 \in V(\hat{C}_1) \cap V(\hat{C}_2)$. In $C_k$ ($k = 1, 2$) there are edge-disjoint paths $W^{(k)}(v_i, v_j)$ for $i, j \in \{1, 2, 3\}$. For each of the three different pairs $(i, j)$ with $i < j$ the graph $\hat{C}_{(i,j)} = W^{(1)}(v_i, v_j) \cup W^{(2)}(v_i, v_j)$ is a circuit. Hence, these three circuits decompose $\hat{C}_1 \cup \hat{C}_2$ in an edge-disjoint way, i.e. $\nu(G) > 2$.

(ii) This is clear by Lemma 1.

For the case $q = 3$ the graph is supposed to be planar. Due to Kuratowski’s Theorem, a graph is planar if and only if it contains no subgraph that is isomorphic to a subdivision of $K_5$ or $K_{3,3}$. Figure 3 shows $K_5$ together with a cycle packing consisting of $q = 3$ cycles. It
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Fig. 3. Nonplanar situation ($K_5$)

cannot be enlarged although $\gamma(K_5) = 6 \geq 2q$. Therefore, the restriction to planar graphs is necessary for the proof of Proposition 4, (ii).

Note, that Proposition 4 has also some implication for a possible construction procedure of $\mathcal{Z}^*$ for a more general case.

Let $G$ be arbitrary and $\hat{\mathcal{Z}} = \{\hat{C}_1, \hat{C}_2, \ldots, \hat{C}_s\}$ be a collection of edge-disjoint circuits in $G$. Obviously $\nu(G) \geq |\hat{\mathcal{Z}}|$.

The following three conditions (I), (II) and (III) on the numbers of vertices of the circuits in $\hat{\mathcal{Z}}$ allow, if satisfied, an augmentation of $|\hat{\mathcal{Z}}|$ to be carried out:

(I) There is $C_i = (V_i, E_i) \in \hat{\mathcal{Z}}$ such that $|V_i| \leq |E_i| - 1$.

(II) There are $C_i = (V_i, E_i), C_j = (V_j, E_j) \in \hat{\mathcal{Z}}$ with $i \neq j$ such that $|V_i \cap V_j| \geq 3$.

(III) There are $C_i = (V_i, E_i), C_j = (V_j, E_j), C_k = (V_k, E_k) \in \hat{\mathcal{Z}}$ with $i, j, k$ pairwise different such that $|V_i \cap V_j| + |(V_i \cup V_j) \cap V_k| \geq 5$ whose union is not homeomorphic to the complete graph $K_5$.

The way of proving Proposition 4, (i) then shows that (I) and (II) are just reformulations of the conditions for the case $q = 1$ and $q = 2$, respectively.

If (III) holds and if (one or two of) the related circuits $C_i, C_j$ and $C_k$ do not already satisfy (I) or (II), we have situations as illustrated in Figure 4. This illustration indicates how to enlarge the given set $\hat{\mathcal{Z}}$ of circuits, obtaining $\hat{\mathcal{Z}}'$.

Fig. 4. Possible situations a) and b) if (III) is satisfied
This leads to the following proposition.

**Proposition 5.** For $G = (V, E)$ let $\hat{Z} = \{\hat{C}_1, \hat{C}_2, \ldots, \hat{C}_s\}$ a set of edge-disjoint circuits in $G$. If one of the conditions (I), (II) or (III) is satisfied, then there is a set $\hat{Z}' = \{\hat{C}'_1, \hat{C}'_2, \ldots, \hat{C}'_{s+1}\}$ of edge-disjoint circuits in $G$; i.e. $\nu(G) \geq s + 1$.

Observe that condition (III) allows $\bar{G} = C_i \cup C_j \cup C_k$ to be a subdivision of an Eulerian supergraph of $K_{3,3}$. Although $\bar{G}$ is not planar it can be decomposed into 4 edge-disjoint circuits. This case is illustrated in Figure 5 (with $\{a, d, v\}$ and $\{b, c, u\}$ being the two vertex sets for the bipartition of $K_{3,3}$). Note that vertex $u \in V(C_j \cap C_k)$ is crucial for this example: if, instead of $u$, there were $u' \in V(C_j)$ and $u'' \in V(C_k)$ two distinct vertices we would face situation a) illustrated in Figure 4.

![Fig. 5. Nonplanar situation ($K_{3,3}$)](image)

Hence, planarity is important to prove the case $q = 3$ in Proposition 4 but for a possible algorithm condition (III) covers more cases in which a larger cycle packing can be found.

The relation among the three conditions will become clear if we consider so called cycle exchanges, that replace two circuits $\hat{C}_i$ and $\hat{C}_j$ by two different ones $\hat{C}_i$ and $\hat{C}_j$. More detailed, starting from $\hat{Z} = \{\hat{C}_1, \ldots, \hat{C}_s\}$ we say that $\check{Z} = \{\check{C}_1, \ldots, \check{C}_s\}$ is generated from $\hat{Z}$ by a cycle exchange if there are circuits $\hat{C}_i, \hat{C}_j \in \hat{Z}$ and $\check{C}_i, \check{C}_j \in \check{Z}$ such that $\hat{C}_i \cup \hat{C}_j = \check{C}_i \cup \check{C}_j$ and $|V(\hat{C}_i) \cap V(\hat{C}_j)| \geq 2$ and $\check{Z} \setminus \{\check{C}_i, \check{C}_j\} = \hat{Z} \setminus \{\hat{C}_i, \hat{C}_j\}$. Such a cycle exchange is illustrated in Figure 6.

![Fig. 6. Generating two cycles by a cycle exchange](image)

Using cycle exchanges we observe the following facts.

(i) If condition (III) holds for some triple of circuits in $\hat{Z}$, there exists a set of edge-disjoint circuits $\check{Z}$ generated by a cycle exchange from $\hat{Z}$ in which condition (II) is satisfied by a pair of circuits.

(ii) If condition (II) holds for some pair of circuits in $\hat{Z}$, there exists a set of edge-disjoint circuits $\check{Z}$ generated by a cycle exchange from $\hat{Z}$ in which condition (I) is satisfied by a circuit.

Conditions (I) to (III) are not necessary for an enlargement of $\hat{Z}$. But the subsequent lemma states that if (I) and (II) are not satisfied, it is not possible to rearrange edges of a pair of circuits in $\hat{Z}$ in such a way that a larger set $\hat{Z}'$ is obtained.
Lemma 2. Let $G$ be arbitrary and $\hat{G} = \{\hat{C}_1, \ldots, \hat{C}_s\}$ be a maximal set of edge-disjoint circuits in $G$. If none of the conditions (I) and (II) is satisfied, there is no pair of circuits $\hat{C}_i, \hat{C}_j \in \hat{G}$ such that $\nu(\hat{C}_i \cup \hat{C}_j) \geq 3$.

3. Cycle packing number and cyclomatic number

In the following we will focus on relations between $\nu(G)$ and the cyclomatic number $\gamma(G)$. We will characterize the subclass of graphs for which both numbers differ by 1.

The case $\gamma(G) = \nu(G)$ is studied e.g. in [33], [13] and characterizes the class of cactus graphs.

Obviously the following lemma holds.

Lemma 3. For an arbitrary graph $G$ we have

(i) $\gamma(G) - \nu(G) \geq 0$

(ii) If $\gamma(\tilde{G}) - \nu(\tilde{G}) = k$ for at least on subgraph $\tilde{G} \subseteq G$, then $\gamma(G) - \nu(G) \geq k$.

Proof. (i) Every cycle of a maximum cycle packing $\mathcal{Z}^*$ of $G$ contains an edge not belonging to a given maximal spanning forest $F$ of $G$. Since cycles of $\mathcal{Z}^*$ are pairwise edge-disjoint each such edge gives rise to a different elementary cycle with respect to $F$.

(ii) Let $G' = G \backslash \{x\}$ where $x$ is an isolated vertex or an edge of $G$. If $x$ is a vertex or a bridge, then $\nu(G') = \nu(G)$ and $\gamma(G') = \gamma(G)$ and, therefore, $\gamma(G') - \nu(G') = \gamma(G) - \nu(G)$. If $x$ is an edge, but not a bridge, then $\nu(G') \geq \nu(G) - 1$ and $\gamma(G') = \gamma(G) - 1$. In this case $\gamma(G') - \nu(G') \leq \gamma(G) - \nu(G)$. Since the subgraph $\tilde{G}$ with $k := \gamma(\tilde{G}) - \nu(\tilde{G}) \geq 0$ is obtained from $G$ by successively deleting the edges from $E(G) \setminus E(\tilde{G})$ and (possibly) deleting isolated vertices respectively, we get $k = \gamma(\tilde{G}) - \nu(\tilde{G}) \leq \gamma(G) - \nu(G)$.

Remark 1. Let $G$ be a connected graph with $\nu(G) = \gamma(G)$, then Lemma 3 induces, that this equality must hold for all subgraphs $\tilde{G} \subsetneq G$. The class of connected graphs which satisfy this property are so called cactus graphs, i.e. connected graphs in which every edge is part of at most one cycle (see [33]). They were introduced in [23] as Husimi trees. If $B_1, \ldots, B_k$ is a block decomposition of $G$, then Proposition 1 (vii) leads to the characterization that $\nu(G) = \gamma(G)$ if and only if $B_i$ is a cycle or an edge for $i = 1, \ldots, k$ (see [10]).

For our next observations we investigate a class $\mathcal{B}$ of graphs, whereas a graph $H$ belongs to $\mathcal{B}$ if and only if $H$ is 2-vertex-connected and one of the following conditions is satisfied:

1. $H$ is a subdivision of a graph consisting either of 3 or 4 different edges connecting the same two vertices
2. There exist $k \geq 3$ cycles $C_1, \ldots, C_k$ in $H$ with
   - $|V(C_i) \cap V(C_j)| \leq 1$ for all $i \neq j$,
   - $\{v \in V(C_i) \mid d_H(v) > 2\} = \{v_i, \bar{v}_i\}$ and $(v_i, \bar{v}_i) \in E(H)$ if and only if $(v_i, \bar{v}_i) \in E(C_i)$,
   - the graph induced by the edges $E(H) \setminus \bigcup_{i=1}^k E(C_i)) \cup \bigcup_{i=1}^k (v_i, \bar{v}_i)$ is a cycle.

In order to get an idea of how a graph $G \in \mathcal{B}$ may look like, consider the following construction principle:

Take a cycle $C$ of length $l \geq 2$ and choose $k \geq 1$ different edges $e_i = (v_i, \bar{v}_i) \in E(C)$, $i = 1, \ldots, k$. After replacing the edges $e_i$ by different cycles $C_i$ for which $V(C_i) \cap V(C) = \{v_i, \bar{v}_i\}$ and $(V(C_i) \setminus \{v_i, \bar{v}_i\}) \cap V(C_j) = \emptyset$ for all $i \neq j$ holds one obtains a graph $G$. 
Graphs which can be obtained in this way belong to $\mathcal{B}$. An example of a graph of this class is given in the following Figure 7.

Moreover,

**Lemma 4.** Every graph $G \in \mathcal{B}$ can be constructed from some cycle $C$ of length $l \geq 2$ by the above stated construction principle.

**Proof.** Let $G \in \mathcal{B}$ satisfying condition 1. If $G$ is a subdivision of a graph $G'$, that contains 3 different edges connecting the same two vertices, say $a$ and $b$, then $G$ consists of 3 paths with endvertices $a$ and $b$ of lengths $l_1$, $l_2$, $l_3$. Let $C$ be a cycle of length $l_1 + 1$ and choose an edge $e = (u, w) \in E(C)$. Now replace $e$ by a cycle $C_1$ of length $l_1 + l_2$ where $u \in V(C_1)$ and $w \in V(C_1)$ divide $C_1$ into two different paths of lengths $l_1$ and $l_2$ respectively. Obviously this graph is isomorphic to $G$.

If $G'$ consists of 4 different edges connecting $a$ and $b$, then $G$ consists of 4 internal vertex-disjoint paths with common endvertices $a$ and $b$ of lengths $l_1$, $l_2$, $l_3$, and $l_4$. Let $C = \{e_1, e_2\} = \{(a, b), (a, b)\}$. Replace $e_1$ and $e_2$ by two different cycles $C_1$ and $C_2$ of length $l_1 + l_2$ and $l_3 + l_4$ respectively in the predescribed way $(V(C_i) \cap V(C) = \{a, b\}$ for $i = 1, 2$ and $(V(C_1) \setminus \{a, b\}) \cap V(C_2) = \emptyset)$. The resulting graph is obviously isomorphic to $G$ then.

If $G \in \mathcal{B}$ satisfies condition 2, let $C$ be the cycle induced by the edge-set $(E(G) \setminus \bigcup_{i=1}^{k} E(C_i)) \cup \bigcup_{i=1}^{k} (v_i, \bar{v}_i)$ with $C_i$, $v_i$, and $\bar{v}_i$ as given in the definition and $e_i = (v_i, \bar{v}_i)$ for $i = 1, \ldots, k$. Replacing the edges $e_i$ successively by the cycles $C_i$ such that the vertices $v_i, \bar{v}_i$ divide the cycle $C_i$ in appropriate paths, this constructs $G$ from the cycle $C$ in the mentioned way.

**Lemma 5.** Let $G = (V, E)$ be a 2-vertex-connected graph. Then $\gamma(G) = \nu(G) + 1$ holds if and only if $G \in \mathcal{B}$.

**Proof.** Due to the fact, that every $G \in \mathcal{B}$ can be obtained using the construction principle above and those graphs resulting from the construction principle are 2-vertex-connected and satisfy $\gamma(G) = \nu(G) + 1$, we immediately see the ”if”-case.

So, consider a 2-vertex-connected graph $G$ with $\gamma(G) = \nu(G) + 1$. Note, that for every pair of vertices in $G$ there are two internally vertex-disjoint paths connecting them in $G$. Therefore every edge $e \in E$ is part of at least one cycle in $G$. If $\nu(G) = 1$ we immediately see, that $G$ is a subdivision of a graph consisting only of 3 parallel edges. If $\nu(G) = 2$ and $E(G) = E(C_1) \cup E(C_2)$ for two cycles $C_1, C_2$ we have $G$ being a subdivision of a graph.
consisting only of 4 parallel edges. Therefore, let \( \nu(G) = 2 \) with \( E(G) \supseteq E(C_1) \cup E(C_2) \) or \( \nu(G) \geq 3 \).

For every cycle \( C_i \) of a maximum cycle packing of \( G \) choose an arbitrary edge \( e_i = (u_i, w_i) \in E(C_i) \) and define \( G' = G \setminus \{ e_i \mid i = 1, \ldots, \nu(G) \} \). Then \( \gamma(G') = \gamma(G) - \nu(G) = 1 \), i.e. in \( G' \) there is exactly one cycle \( \bar{C} \).

**Claim 1:** For \( e = (u, w) \in \bar{E} \) with \( e \notin \bigcup_{i=1}^{\nu(G)} E(C_i) \) we have \( e \in E(\bar{C}) \).

Proof of Claim 1: \( e \) is an edge of at least one cycle \( \bar{C} \) in \( G \). We replace \( e_i \) by the path \( P_i := C_i \setminus e_i \) for every \( i \in \{ j \mid e_j \in E(\bar{C}) \} \) which gives rise to a closed walk \( \bar{C}' \) in \( G' \). In \( \bar{C}' \setminus \{ e \} \) we find a path \( P \) connecting \( u \) and \( w \) and \( \bar{E}(P) \cup \{ e \} \) induces a cycle in \( G' \). Since \( \bar{C} \) is the only cycle in \( G' \) the claim follows.

**Claim 2:** \( E(C_i) \cap E(\bar{C}) \neq \emptyset \) for all \( i = 1, \ldots, \nu(G) \).

Proof of Claim 2: Assume there is \( i \) with \( E(C_i) \cap E(\bar{C}) = \emptyset \). In this case \( |V(C_i) \cap V(\bar{C})| \leq 1 \), since otherwise let \( \nu', \nu'' \in V(C_i) \cap V(\bar{C}) \) be two different vertices. The subgraph \( S \) of \( G' \) that consists of the cycle \( \bar{C} \) and the path in \( C_i \setminus e_i \) with endvertices \( \nu' \) and \( \nu'' \) has a cyclomatic number \( \gamma(S) > 1 \) which is impossible by Lemma 3.

Let \( v^* \in V(C_i) \setminus V(\bar{C}) \) and \( \bar{v} \in V(\bar{C}) \setminus V(C_i) \) be vertices of \( G \). There are two internally vertex-disjoint paths in \( G \) connecting \( v^* \) and \( \bar{v} \) denoted by \( \bar{P}^{(1)} \) and \( \bar{P}^{(2)} \). Let \( v_1 \) be the last vertex of \( V(C_i) \) in \( \bar{P}^{(1)} \) starting from \( v^* \) and \( v_2 \) the last one of \( V(C_i) \) in \( \bar{P}^{(2)} \) respectively.

Note, that \( v_1, v_2 \) and \( \bar{v} \) are pairwise different vertices and \( \{ v_1, v_2 \} \not\subseteq \bar{V}(\bar{C}) \), i.e. w.l.o.g.

Since \( v_1 \notin V(\bar{C}) \). In \( C_i \setminus e_i \) there is a path \( \bar{P}^{(1)} \) between \( v_1 \) and \( v_2 \). Let \( \bar{P}^{(1)} \) be the path in \( \bar{P}^{(1)} \) connecting \( v_1 \) and \( \bar{v} \) and \( \bar{P}^{(2)} \) the path in \( \bar{P}^{(2)} \) starting in \( v_2 \) and ending in \( \bar{v} \) respectively.

Concatenating \( \bar{P}^{(1)} \), \( \bar{P}^{(1)} \) and \( \bar{P}^{(2)} \) leads to a cycle \( C \) in \( G \). Obviously, \( \bar{P}^{(1)} \subseteq G' \) and if neither \( \bar{P}^{(1)} \) nor \( \bar{P}^{(2)} \) contain an edge \( e_j \) for some \( j \), then \( C \subseteq G' \), i.e. \( \gamma(G') > 1 \).

If \( J := \{ j \mid e_j \in \bar{P}^{(1)} \cup \bar{P}^{(2)} \} \neq \emptyset \) we can find a cycle \( \bar{C} \) in \( G \) with \( \bar{E}(\bar{P}^{(1)}) \cup \bar{E}(\bar{P}^{(2)}) \) and \( \bar{J} := \{ j \mid e_j \notin \bar{E}(\bar{C}) \} \subseteq J \):

Let \( j \in J \) and \( v^*_j \) and \( v^*_j \) be the first and last vertex in \( C \setminus \bar{P}^{(1)} \) from \( v_1 \) to \( v_2 \) with \( v^*_j, v^*_j \in V(C_j) \). If we replace the path between \( v^*_j \) and \( v^*_j \) in \( C \setminus \bar{P}^{(1)} \) by the unique path in \( C_j \setminus e_j \) between those two vertices we find a cycle \( \bar{C} \) with \( e_j \notin \bar{E}(\bar{C}) \) and \( \bar{J} \subseteq J \).

By a successive replacement of edges \( e_j \) in \( \bar{C} \) we finally get a cycle \( \bar{C}^* \) with \( \bar{E}(\bar{P}^{(1)}) \cup \bar{E}(\bar{P}^{(2)}) \) and \( e_k \notin \bar{E}(\bar{C}^*) \) for all \( k = 1, \ldots, \nu(G) \), i.e. \( \gamma(G^*) > 1 \). This completes the proof of Claim 2.

By Claim 2 every path \( C_i \setminus e_i \) from \( u_i \) to \( w_i \) contains a first vertex \( v_i \in V(\bar{C}) \) and a last vertex \( \bar{v}_i \in V(\bar{C}) \) with \( v_i \neq \bar{v}_i \). The edges of paths from \( u_i \) to \( v_i \) and from \( \bar{v}_i \) to \( w_i \) in \( C_i \setminus e_i \) do not belong to the cycle \( \bar{C} \). Since \( \gamma(G^*) = 1 \) the paths from \( v_i \) to \( \bar{v}_i \) in \( C_i \setminus e_i \) are subgraphs of \( \bar{C} \).

Therefore we see that \( \{ v \in V(C_i) \mid d_G(v) > 2 \} = \{ v_i, \bar{v}_i \} \) and \( \{ v_i, \bar{v}_i \} \subseteq E(G) \) if and only if \( (v_i, \bar{v}_i) \subseteq E(C_i) \).

We conclude the proof by showing \( |V(C_i) \cap V(C_j)| \leq 1 \) for all \( 1 \leq i < j \leq \nu(G) \) for \( \nu(G) \geq 3 \):

If \( |V(C_i) \cap V(C_j*)| \geq 2 \) for some pair of different indices \( i^* \neq j^* \), then \( \gamma((C_i^* \setminus e_{i^*}) \cup (C_j^* \setminus e_{j^*})) \geq 1 \). Since \( \nu(G) \geq 3 \) there is a cycle \( C_k^* \), with \( k^* \notin \{ i^*, j^* \} \) with \( E(C_k^*) \cup E(\bar{C}) \neq \emptyset \) by Claim 2. Therefore \( \bar{C} \not\subseteq \{(C_i^* \setminus e_{i^*}) \cup (C_j^* \setminus e_{j^*}) \} \) and by Lemma 3 we have \( \gamma(G') > 1 \) which is a contradiction. Thus \( G \in B \).

Using these facts we can now prove the following theorem.

**Theorem 1.** Let \( G \) be a graph. Then \( \gamma(G) = \nu(G) + 1 \) if and only if \( G \) has a decomposition into blocks of which all but one are either cycles or edges. The remaining block is a graph in \( B \).
Proof. Let $G$ be a graph satisfying the block decomposition stated in the theorem. For blocks $B$ which are edges or cycles we have $\gamma(B) = \nu(B)$. For the remaining block $\bar{B}$ we have $\gamma(\bar{B}) = \nu(\bar{B}) + 1$ according to the previous lemma. Due to Proposition 1, (vii) the equality $\gamma(G) = \nu(G) + 1$ holds.

Now, consider a graph $G$ with $\gamma(G) = \nu(G) + 1$. Using Lemma 3 we know there is no block $B$ in $G$ with $\gamma(B) = \nu(B) + k$ for a $k \geq 2$. Due to Proposition 1, (vii) there cannot be two or more different blocks in $G$ with $\gamma(B) = \nu(B) + 1$. Therefore there is exactly one block $B$ for which $\gamma(B) = \nu(B) + 1$ is true. As proved in the previous lemma we have $B \in \mathcal{B}$. $G \setminus E(B)$ consists of components that are cactus graphs and therefore all of its blocks are either cycles or edges.

A further characterization for connected graphs $G$ with $\gamma(G) = \nu(G) + 1$ immediately follows.

**Corollary 1.** Let $G$ be a connected graph. Then $\gamma(G) = \nu(G) + 1$ if and only if there is a cactus graph $G'$ with $v_1, v_2 \in V(G')$ being vertices of different blocks in $G'$ which are connected by at least two different paths in $G'$ and $G$ is given by the contraction $G' / \{v_1, v_2\}$.

![Graph](image.png)

**Fig. 8.** Graph $G$ (black and grey edges and vertices) with $\gamma(G) = \nu(G) + 1$ and 2-vertex-connected subgraph $G'$ (black edges and vertices only) with $G' \in \mathcal{B}$

References


On a relation between the cycle packing number and the cyclomatic number of a graph


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