Public goods, participation constraints, and democracy:  
A possibility theorem*

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Abstract

It is well known that ex post e¢ cient mechanisms for the provision of indivisible public goods are not interim individually rational. However, the corresponding literature assumes that agents who veto a mechanism can enforce a situation in which the public good is never provided. This paper instead considers majority voting with uniform cost-sharing as the relevant status quo. Efficient mechanisms may then exist, which also satisfy all agents' interim participation constraints. In this case, ex post inefficient voting mechanisms can be replaced by efficient ones without reducing any individual's expected utility. Intuitively, agents with a low willingness to pay have to contribute more under majority rule than under an efficient mechanism with a balanced budget. Although this possibility theorem is not universal in the sense of Schweizer (Games and Economic Behavior, 2006), an asymptotic possibility is obtained for certain type distributions.

Keywords: Public goods provision, ex post efficiency, participation constraints, majority voting.

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1 Introduction

Mechanism design theory has produced some very useful procedures for collective decision making. Important examples are the second price auction and, as generalizations, the Vickrey Clarke Groves (VCG) and the Arrow, d’Aspremont, Gérard-Varet (AGV) mechanisms. While the second price auction is frequently used for the allocation of indivisible private goods, AGV mechanisms for indivisible public projects can rarely be found in practice. A standard argument why such mechanisms are not used for decisions about public projects is that they are too costly for some citizens. Indeed, it is a well known result that ex post efficient mechanisms for the provision of indivisible public goods are generally not interim individually rational.\footnote{See Güth and Hellwig (1986), Mas Colell, Whinston and Green (1995, Chapter 23), Mailath and Postlewaite (1998), and Hellwig (2003), as well as Krishna and Perry (2000) and Schweizer (2005) for simplified proofs and generalizations. A method to conveniently calculate the budget deficit under an ex-post efficient mechanism with binding interim participation constraints can be found in Krishna and Perry (2000).} Under a AGV mechanism, agents who do not care about the public good may have to contribute to finance its provision or, alternatively, they have to pay for the externality that they generate when the good is not provided.

However, the corresponding literature assumes that agents who veto a mechanism can enforce a situation in which the public good is never provided. Today, in most economies public goods are provided, and mechanisms for the provision of public goods are used. Therefore, the question whether existing mechanisms can be replaced by ex post efficient ones can not be answered on the grounds of the existing literature.

This paper studies potential improvements of decision procedures for public projects. It investigates whether an ex post inefficient voting mechanism can be replaced by an ex post efficient one without reducing any individual’s expected payoff. Expected payoffs are evaluated at the interim stage, where individuals already hold private information about their willingness to pay for the public project. We formally characterize those cases in which, at the interim stage, all agents unanimously prefer an ex post efficient mechanism to simple majority voting.

In practice, decisions about public goods are based frequently on some variant of a voting mechanism. Such voting mechanisms also violate some agents’ participation constraints when the cost of the public good has to be shared by all agents. In many cases, the procurement of a public good will either lead to an increase of the tax burden for all individuals or to a reduction of the budget for future expenditure. This budget effect creates an opportunity cost of public spending. In this paper, we assume that the cost of the public good is shared equally among all agents. Agents decide whether they are willing to replace an existing voting procedure by an efficient mechanism after learning their own type, but before playing the voting game.

We consider a general class of voting mechanisms. A voting mechanism asks all players for a binary signal: “yes” or “no”. It then monotonically maps all agents’ votes into a collective decision. Monotonicity means that it becomes (weakly) more likely that the public good is provided if more voters are in favor of provision. Standard voting mechanisms fix the probability of implementation
at 1 (0) if there is a majority (minority) in favor of it. We consider a somewhat more general class of mechanisms, that permits random outcomes for votes that are close to a tie. This captures the idea that actual political outcomes need not be perfectly linked to voters’ intentions and may also to a certain extent be influenced by other factors such as political influence or corruption.

Our main result is a characterization of (expected) externality functions which lead to a possibility result in the sense that an ex post efficient balanced-budget and interim rational mechanism exists. We provide two examples for type distributions that fulfill the condition we derive. In the case where the net willingness to pay is normally distributed, we prove that the possibility result holds for both small and large population sizes. This is a stark contrast with what is generally understood in the literature (Mailath and Postlewaite, 1990); the probability of public good provision converges to zero as the population size grows. Intuitively, the reason for our possibility result is that agents with a low willingness to pay have to contribute more under majority rule than under an efficient mechanism with a balanced budget. Interestingly, the possibility depends on the parity of the population size; for both normal and uniform distribution, the possibility result is obtained if and only if the population size is even. We provide further explanation in Section 4.

In section 2, we introduce the general model. Section 3 has the main result and section 4 provides the examples.\footnote{Note that, however, our possibility theorem is not universal in the sense of Schweizer (2005). Universality would require that the participation constraints can be fulfilled independently of the distribution of types. Note also that Schweizer’s convenient method of testing for a universal possibility - or impossibility - theorem cannot be applied in the present context. This is because, with majority voting as a status quo, the type with the smallest interim gain from a new mechanism need not be located at the boundary of the type space.}

**Related literature**

The present paper contributes to the literature on the ex post efficient provision of public goods, including in particular the papers by Güth and Hellwig (1986), Mailath and Postlewaite (1990), and Hellwig (2003). Güth and Hellwig (1986) prove an impossibility result à la Myerson Satterthwaite (1983) in the public goods context. Mailath and Postlewaite (1990) consider the case with a large number of agents when the size of the project varies with the size of the population. They find that the second best surplus of the economy goes to zero when the population size grows. Hellwig (2003) instead considers the case where the size of the project does not grow unboundedly. Positive (possibility) results can be found in papers which assume correlated types such as McAfee and Reny (1985). However, if one introduces stricter equilibrium concepts such as Bergemann and Morris’s (2005) robustness criterion, the results are again negative (Bierbrauer and Hellwig, 2008).

Another way of guaranteeing participation in efficient mechanisms which has recently been studied in the literature, is the bundling of various decisions (Casella, 2005, Fang and Norman 2005, 2008, and Jackson and Sonnenschein, 2007).

The present analysis follows closely an idea that has been developed in Cramton, Gibbons and Klemperer (1987). Their paper shows in a trade context that initial conditions are key in determining whether or not a possibility or an impossibility theorem à la Myerson and Satterthwaite
In another related paper, Schmitz (2002) also considers the case of a public project. He shows that a possibility theorem may be obtained when the status quo is a stochastic decision with an exogenously fixed provision probability. In such a case, an agent’s interim status quo utility is linear in his type while the VCG interim utility is strictly convex. A possibility theorem holds because the VCG interim utility of the indifferent type (i.e. the type whose gross willingness to pay equals the average cost) is positive. This agent’s payoff equals the average externality that he generates. As will become clear below, the status quo utility in the present paper is piecewise linear instead, which makes the formal analysis more complicated.

2 The Model

Suppose that \( n \) individuals jointly decide whether to provide an indivisible public good. The decision is denoted by \( x \in \{0,1\} \). Let \( \theta_i \) be the valuation of individual \( i \), and let \( t_i \) be the transfer to \( i \). The payoff of individual \( i \) is quasi-linear:

\[
 u_i(x, \theta_i, t) = \theta_i x + t_i.
\]

A planner can procure the good at a known cost of \( C = nc \). Let \( \tilde{\theta}_i := \theta_i - c \) denote \( i \)'s net willingness to pay. We assume that \( \tilde{\theta}_i \) is private information, independently and identically drawn from a common prior \( F(\cdot) \). Let \( \Theta \) be the support of the distribution. We assume that \( F \) is continuous and \( \Theta \) is connected.

**Timing**

First, each individual receives the private information. Then, all individuals decide whether to participate in the provision mechanism for the public good or not. If somebody refuses, the status quo game is played. If all participate, they play the direct revelation game with the d’Aspremont, Gérard-Varet (AGV) mechanism.

**Status Quo**

In the status quo, the majority voting mechanism is used. All individuals simultaneously decide whether to vote against or for provision; \( \xi_i \in \{0,1\} \). Let \( \xi = (\xi_1, \cdots, \xi_n) \). The decision rule is \( g : \{0,1\}^n \rightarrow \Delta(\{0,1\}), \) where \( g(\xi) \) is the probability that the public good is provided. A super-majority rule is described as:

\[
g(\xi) = 1 \left( \sum_i \xi_i > m \right) + p \cdot 1 \left( \sum_i \xi_i = m \right)
\]

where \( 1(\cdot) \) is the characteristic function. The simple majority rule with uniform tie-breaking is given by \( m = n/2 \) and \( p = 1/2 \). When the public good is provided, the cost is shared equally, that is, each individual pays \( c \).
The AGV mechanism

The AGV mechanism is a direct mechanism, which satisfies ex post efficiency, incentive compatibility, and interim balanced-budget condition. However, it is well-known that the interim participation constraint is not satisfied in general. Since we discuss the interim participation constraint here, we focus on the interim balanced-budget condition.

3 Main Theorem

In this section, we derive the interim expected utility both under the AGV mechanism and under majority rule. Let \( \pi(\tilde{\theta}_i) \) be the interim probability of provision of the public good in the AGV mechanism. By efficiency,

\[
\pi(\tilde{\theta}_i) := \Pr_{\tilde{\theta}_i} \left[ \tilde{\theta}_i + \sum_{j \neq i} \tilde{\theta}_j > 0 \right].
\]

Net externality

Let \( s_{-i} \) be the sum of other individuals’ net willingness to pay; \( s_{-i} := \sum_{j \neq i} \tilde{\theta}_j \), and let \( F_{-i}(\cdot) \) be the cdf of \( s_{-i} \). We define the net externality as:

\[
h(\tilde{\theta}_i) := \int_{-\tilde{\theta}_i}^{\tilde{\theta}_i} s dF_{-i}(s).
\]

This definition is compatible with the standard definition of the externality. To see that, suppose \( \tilde{\theta}_i < 0 \). Then, the presence of individual \( i \) makes a difference in the ex post efficient decision if and only if \( s_{-i} + \tilde{\theta}_i < 0 < s_{-i} \), which is equivalent to \( 0 < s_{-i} < -\tilde{\theta}_i \). In such a case, the difference in the sum of the welfare of other individuals from the provision of the public good is \( s_{-i} \). Therefore, \( i \)'s net externality is

\[
\int_{-\tilde{\theta}_i}^{\tilde{\theta}_i} s dF_{-i}(s).
\]

The same expression is obtained for the case \( \tilde{\theta}_i > 0 \) as well.

Now, define \( r \) as the ex ante expected net externality:

\[
r := E_{\tilde{\theta}_i} \left[ h(\tilde{\theta}_i) \right].
\]

Let \( U(\tilde{\theta}_i) \) be the interim expected payoff of individual \( i \) obtained in the AGV mechanism. Then, by definition, we have:

\[
U(\tilde{\theta}_i) = \tilde{\theta}_i \pi(\tilde{\theta}_i) - h(\tilde{\theta}_i) + r.
\]

Majority Rule
Let \( M(\tilde{\theta}_i) \) be the interim expected payoff of individual \( i \) under majority rule (2). It is straightforward to show that in any dominant strategy,

\[
\xi_i = \begin{cases} 
1 & \text{if } \tilde{\theta}_i > 0 \\
0 & \text{if } \tilde{\theta}_i < 0 
\end{cases}
\]

Hence, \( M \) is piecewise linear; \( \exists \alpha^+, \alpha^- \in \mathbb{R}^+ \) such that

\[
M(\tilde{\theta}_i) = \begin{cases} 
\alpha^+ \tilde{\theta}_i & \text{if } \tilde{\theta}_i > 0 \\
\alpha^- \tilde{\theta}_i & \text{if } \tilde{\theta}_i < 0 
\end{cases}
\]

We often consider the case where \( \tilde{\theta}_i \) is symmetrically distributed; \( F(\tilde{\theta}_i) + F(-\tilde{\theta}_i) = 1 \). We can give an explicit formula of \( M(\tilde{\theta}_i) \) for the simple majority rule. In order to show explicitly that the values depend on the society size \( n \), we add subscripts to the coefficients.

**Lemma 1** Suppose that \( \tilde{\theta}_i \) is symmetrically distributed, and consider the simple majority rule: \( m = n/2, \ p = 1/2 \). If \( n \) is even,

\[
\alpha^+_n = \frac{1}{2} + \frac{1}{2^n} \left( \frac{n-1}{n/2-1} \right), \quad \alpha^-_n = \frac{1}{2} - \frac{1}{2^n} \left( \frac{n-1}{n/2-1} \right).
\]

If \( n \) is odd, \( \alpha^+_n = \alpha^-_{n-1}, \ \alpha^-_n = \alpha^+_{n-1} \).

**Proof.** See Appendix. \( \blacksquare \)

Notice that when \( n \) is odd, the probabilities of acceptance in the simple majority rule (conditional on \( i \)'s vote) are the same in the committee of size \( n \) and of size \( n - 1 \). This is because the \( n \)-th vote does not make any difference except for the case where the \( n \)-th vote breaks the tie (in the other cases, the difference of yes votes and no votes is 2 or more among the other \( n - 1 \) voters, hence the \( n \)-th voter cannot make any difference). But in the committee of size \( n - 1 \), the tie is broken with an equal probability (\( p = 1/2 \)) anyway. The \( n \)-th voter plays the exactly same role as a fair coin toss.

Now, define two sets of types as follows:

\[
\Theta^+ = \begin{cases} 
\{ \theta | \pi(\theta) = \alpha^+ \} & \text{if } \exists \theta \in \Theta \cap \mathbb{R}^+ \text{ s.t. } \pi(\theta) = \alpha^+ \\
\{0\} & \text{if } \forall \theta \in \Theta \cap \mathbb{R}^+, \pi(\theta) > \alpha^+ \\
\{\sup \Theta\} & \text{if } \forall \theta \in \Theta \cap \mathbb{R}^+, \pi(\theta) < \alpha^+ 
\end{cases}
\]

\[
\Theta^- = \begin{cases} 
\{ \theta | \pi(\theta) = \alpha^- \} & \text{if } \exists \theta \in \Theta \cap \mathbb{R}^- \text{ s.t. } \pi(\theta) = \alpha^- \\
\{0\} & \text{if } \forall \theta \in \Theta \cap \mathbb{R}^-, \pi(\theta) < \alpha^- \\
\{\inf \Theta\} & \text{if } \forall \theta \in \Theta \cap \mathbb{R}^-, \pi(\theta) > \alpha^- 
\end{cases}
\]

Note that both sets are nonempty. Moreover, both sets contain only one element if \( \pi(\theta) \) is strictly increasing. As we will see below, the types in \( \Theta^+ \) and \( \Theta^- \) are the most reluctant to

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\(^3\)Since we assume that the distribution of \( \tilde{\theta}_i \) has no mass point, \( \{ \theta | \pi(\theta) = \alpha^+ \} \) is connected. Therefore, \( \Theta^+ \) and \( \Theta^- \) are well-defined.
participate in the AGV mechanism with majority voting as the status quo when $\tilde{\theta}_i \geq 0$ or $\tilde{\theta}_i \leq 0$ respectively. All elements of $\Theta^+$ and $\Theta^-$ lie in the interior of the type space $\Theta$ if:

$$\inf_{\theta \in \Theta} \pi(\theta) < \alpha^- \text{ and } \sup_{\theta \in \Theta} \pi(\theta) > \alpha^+. \quad (3)$$

Here is our main theorem.

**Theorem 1** Suppose that the status quo is majority voting (2) and let condition (3) hold. The AGV mechanism is (i) ex post efficient, (ii) Bayesian incentive compatible, (iii) ex post balanced-budget and (iv) satisfies the interim participation constraint only if and only if

$$r > \max_{\theta \in \Theta^+ \cup \Theta^-} h(\theta). \quad (4)$$

**Proof.** Since the AGV mechanism satisfies conditions (i)-(iii), it remains to consider condition (iv). Remember that

$$U(\bar{\theta}_i) = \bar{\theta}_i \pi(\bar{\theta}_i) - \bar{\theta}_i h(\bar{\theta}_i) + r.$$

Ex post efficiency and incentive compatibility imply that $U'(\bar{\theta}_i) = \pi(\bar{\theta}_i).$ Since $\pi$ is non-decreasing, $U$ is weakly convex. Since $M(\theta)$ is linear in $\theta \geq 0$ (resp. $\theta \leq 0$), $U(\theta) - M(\theta)$ is weakly convex in $\theta \geq 0$ (resp. $\theta \leq 0$).

We claim that the infimum of $U(\theta) - M(\theta)$ is attained at any $\theta$ in $\Theta^+$ (resp. $\Theta^-$) within the domain of $\theta \in \Theta \cap \mathbb{R}^+$ (resp. $\theta \in \Theta \cap \mathbb{R}^-)$. By definition of $\Theta^+$, we have three cases: (a) if $\exists \theta \in \Theta \cap \mathbb{R}^+$ s.t. $\pi(\theta) = \alpha^+$, then $\pi(\theta^+) = \alpha^+$ for $\theta^+ \in \Theta^+.$ Since $U'(\theta) = \pi(\theta)$ and $M'(\theta) = \alpha^+$ in $\theta > 0,$ $U'(\theta^+) = M'(\theta^+).$ Since $U(\theta) - M(\theta)$ is weakly convex in $\theta \geq 0,$ the minimum (in $\theta \in \Theta \cap \mathbb{R}^+$) of $U(\theta) - M(\theta)$ is attained at $\theta^+.$ (b) If $\forall \theta \in \Theta \cap \mathbb{R}^+, \pi(\theta) > \alpha^+$, then $U'(\theta) > M'(\theta)$ for $\forall \theta > 0.$ Since $U(\theta) - M(\theta)$ is increasing, the infimum (in $\theta \in \Theta \cap \mathbb{R}^+$) is attained at $\theta = 0$ which is the unique element in $\Theta^+,$ by definition. (c) If $\forall \theta \in \Theta \cap \mathbb{R}^+, \pi(\theta) < \alpha^+$, then $U'(\theta) < M'(\theta)$ for $\forall \theta > 0.$ Since $U(\theta) - M(\theta)$ is decreasing, the infimum (in $\theta \in \Theta \cap \mathbb{R}^+$) is attained at sup $\Theta$, which is the unique element in $\Theta^+,$ by definition.

Now, for $\theta \geq 0,$

$$U(\theta) \geq M(\theta) \Leftrightarrow r \geq h(\theta) + (\alpha^+ - \pi(\theta)) \theta.$$

For cases (a) and (b), $(\alpha^+ - \pi(\theta^+)) \theta^+ = 0$ for $\forall \theta^+ \in \Theta^+.$ Hence, $U(\theta) \geq M(\theta)$ for $\forall \theta \geq 0$ if and only if $r \geq h(\theta^+).$ For case (c), $(\alpha^+ - \pi(\theta^+)) \theta^+ > 0$ and (iv) imples $r > h(\theta^+)$. The proof is analogous for $\theta \leq 0.$

In sum, condition (iv) implies (4) in all cases. If, moreover, condition (3) is satisfied, then both $\Theta^+$ and $\Theta^-$ fall in the cases (a) or (b). Therefore, condition (4) implies (iv). ■

**Remark 1** This is a non-universal result. The possibility depends on the distribution of the types. We discuss more in detail using diverse examples in the next section.

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4Condition (3) is satisfied if the domain $\Theta$ is unbounded, except for uninteresting cases where, for example, $\Pr(\tilde{\theta}_i > 0) = 0$ or 1, which implies $\alpha^- = 0$ or $\alpha^+ = 1$. Even when the domain $\Theta$ is bounded, condition (3) is satisfied in all examples in Section 4.2.
Condition (4) has a straightforward interpretation. An individual which faces the same probability of the decision $x = 1$ under both mechanisms only has to consider the transfer payment under the AGV mechanism. This payment is the difference between the two components $r$ and $h$. The overall expected externality $r$ only depends on the distribution of types. The expected externality of the most reluctant type, $\max_{\theta \in \Theta^+} h(\theta)$, depends both on the distribution of types and on the voting rule.

To see how the majority rule affects the result, consider a given type distribution which fixes $r$, $h(\theta)$ and $U(\theta)$. According to Theorem 1, whether the possibility result is obtained or not depends on the comparison of the following three values: $r$, $\max_{\theta \in \Theta^+} h(\theta)$ and $\max_{\theta \in \Theta^-} h(\theta)$. A change of the majority rule leads to a change of the slopes of the curve $M(\hat{\theta}_i)$, and therefore the location of the values in $\Theta^+$ and $\Theta^-$. A larger majority requirement (a larger $m$ and/or a smaller $p$ in (2)) decreases the probabilities of public good provision, $\alpha^+$ and $\alpha^-$. This reduces the values in $\Theta^+$ because the probability $\pi(\hat{\theta}_i)$ is increasing in $\hat{\theta}_i$. Then, $\max_{\theta \in \Theta^+} h(\theta)$ decreases, because $h(\hat{\theta}_i)$ is increasing in $\hat{\theta}_i \in \mathbb{R}^+$. However, one obtains the opposite effect for negative net valuations $\hat{\theta}_i \in \mathbb{R}^-$. A larger majority requirement reduces $\Theta^-$ in $\Theta^-$, implying an increase of $\max_{\theta \in \Theta^-} h(\theta)$.

With a symmetric distribution of types, the distance $U(\hat{\theta}_i) - M(\hat{\theta}_i)$ is also symmetric around zero. Any deviation from a symmetric voting rule ($m = n/2$) shifts both $\Theta^+$ and $\Theta^-$ in the same direction, implying an increase of either $\max_{\theta \in \Theta^+} h(\theta)$ or $\max_{\theta \in \Theta^-} h(\theta)$. Starting from the symmetric majority rule, which implies $\max_{\theta \in \Theta^+} h(\theta) = \max_{\theta \in \Theta^-} h(\theta)$, such a deviation only makes it harder to satisfy condition (4). Therefore, when types are distributed symmetrically, a symmetric voting rule ($m = n/2$) facilitates the participation of all types. If there is no possibility result for the standard majority rule, there is no possibility result for any majority rule.

Another interpretation of condition (4) concerns the location of the distribution. Consider a distribution with a given shape and variable mean. As the mean increases, the slopes $\alpha^+$ and $\alpha^-$ increase. Again, ceteris paribus, this would shift both $\Theta^+$ and $\Theta^-$ to the left. However, the shape of $h(\theta)$ is also affected. When the mean increases, it becomes less likely that an individual is pivotal under the AGV mechanism. This is why the expected externality eventually decreases and the $U$ curve becomes linear. The interaction of both effects is complex, and we study several examples in detail in the next section.

4 Examples

In this section, we consider several type distributions to see how the possibility result would depend on the distribution of the valuation, as well as the size of the society.

4.1 Normal distribution

Suppose that the random variables $\left(\hat{\theta}_i\right)_{i=1}^n$ are independently drawn from a normal distribution with mean 0, variance $\sigma^2$. We can deduce explicit formulae for $h(\theta)$ and $r$ and use Theorem 1 to
see whether a possibility result is obtained or not. First, note that $\sum_{j \neq i} \tilde{\theta}_j$ is normally distributed with mean 0 and variance $(n - 1) \sigma^2$. By efficiency,

$$\pi \left( \tilde{\theta}_i \right) = \text{Pr} \left[ \sum_{j \neq i} \tilde{\theta}_j > -\tilde{\theta}_i \right] = 1 - \Phi \left( \frac{-\tilde{\theta}_i}{\sqrt{n - 1} \sigma} \right) = \Phi \left( \frac{\tilde{\theta}_i}{\sqrt{n - 1} \sigma} \right),$$

where $\Phi$ is the cdf of the standard normal distribution:

$$\Phi (x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp \left( -\frac{t^2}{2} \right) dt.$$

Now, we are ready to state a possibility result for even $n$.

**Theorem 2** Suppose that the net valuations are drawn independently from a normal distribution with mean 0. For even $n$, the AGV mechanism is (i) ex post efficient, (ii) Bayesian incentive compatible, (iii) ex post balanced-budget, and (iv) satisfies the interim participation constraint when the status quo is the simple majority rule with uniform tie-breaking rule ($m = n/2$, $p = 1/2$).

**Proof.** See Appendix. ■

Notice that this result is independent of the size of the group, $n$. This is a vivid contrast with the results in the literature where negative results are obtained for large $n$ (e.g. Mailath and Postlewaite (1990), Lehrer and Neeman (2000)). An intuitive explanation for the difference is the following: the probability of any individual being pivotal for the social decision converges to zero as the group size $n$ goes to infinity. Therefore, in order to provide an incentive to report the valuation truthfully, an individual with a high valuation cannot be taxed too much more than a low-valuation individual. When the status quo is non-provision, veto power ensures that the payment of a low-valuation individual must be small. When the status quo is a majority rule, the payment of a low-valuation individual can be higher, since the payment would be also high under majority rule with equal share. Therefore, the veto power has a less bite in our model.

Also, note that the result is independent of the standard deviation of the normal distribution. This is because changing $\sigma$ corresponds merely to changing the unit of the payoffs. Since we assume that $\tilde{\theta}_i$ is symmetrically distributed around zero, the result is invariant with respect to the unit.

Interestingly, condition (4) is not satisfied when $n$ is odd. As we have emphasized, the possibility result is not universal. We saw in Lemma 1 that the interim expected payoff is the same in the groups of size $n$ and $n - 1$, when $n$ is odd. However, the expected payoff of the AGV mechanism decreases as the group size becomes large. Numerically, we found that for any odd integer $n$, there exists some $\theta$ such that $U_n (\theta) < M_n (\theta) = M_{n-1} (\theta) < U_{n-1} (\theta)$.\(^5\)

However, numerical examples also show that the probability of $U_{2n'+1} (\theta) > M_{2n'+1} (\theta)$ converges to one as $n'$ goes to infinity. This is a sharp contrast as compared to the case where the status quo is non-provision.

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\(^5\)Note also that using an additional $(1/2, 1/2)$ random vote would lead to a possibility theorem for an odd number of agents. The interim utility is now equal to $M_{n+1} (\theta)$. Since $M_{n+1} (\theta) < U_{n+1} (\theta) < U_{n} (\theta)$, we have a possibility result.
4.2 Uniform distribution

4.2.1 $n=2$

Suppose that the net valuation $\tilde{\theta}_i$ is drawn from a uniform distribution. For $n=2$, we have a possibility theorem.

**Theorem 3** Suppose $n = 2$, and $\tilde{\theta}_i \sim \text{unif}[-a, b]$ with $a > 0$, $b > 0$. There exists a continuum of values $\bar{p} \in [p, \bar{p}]$, such that the AGV mechanism satisfies (i) efficiency, (ii) Bayesian incentive compatibility, (iii) balanced-budget condition, and (iv) the interim participation constraint, where the status quo is a majority rule $m = 1$ with tie-breaking rule $p$. Moreover, if $a = c$ and $b = 1 - c$, both $\bar{p}$ and $\bar{p}$ are strictly decreasing in $c$.

**Proof.** See Appendix. □

Theorem 3 shows that there always exists a monotonic voting mechanism as the status quo such that the AGV mechanism can implement the ex post efficient social choice. Suppose $a = c$ and $b = 1 - c$. Then, $\tilde{\theta}_i \sim \text{unif}[-c, 1 - c]$, hence $\theta_i \sim \text{unif}[0, 1]$. The theorem covers the case where the gross valuation of the public good $\theta_i$ is uniformly distributed over $[0, 1]$, and the per capita production cost is $c \in (0, 1)$. If $c \notin (0, 1)$, then obviously all the individuals agree on the efficient decision, therefore the question is trivial. The theorem implies that, if a society applies more restrictive majority rule mechanisms to more costly projects, in the sense that $p$ decreases in $c$, then it may be possible that for all non-trivial values of $c$ the AGV mechanism can implement a social choice which is ex post efficient and budget-balancing, without violating the participation constraints.
For example, the following are the graphs of $U(\theta)$ and $M(\theta)$, for the case $c = 0.4$. This implies $a = 0.4$ and $b = 0.6$, that is, the gross valuation of the public good is $\theta_i \sim \text{unif}[-0.4, 0.6]$. The solid curve is the expected payoff $U(\theta)$ from the budget-balancing AGV rule, the dashed (resp. dotted) line is the expected payoff from majority rule with the highest (resp. lowest) $p$ for which the possibility theorem holds. The possibility theorem holds if and only if the $p$ is chosen between these two values.

![Graph of $U(\theta)$, $c = 0.4$, $\theta \sim \text{unif}[0, 1]$](image)

The following graph shows the range of the tie-breaking rule $p$ for which the possibility theorem holds. We change per capita cost $c$ from 0 to 1, holding $\bar{\theta}_i \sim \text{unif}[-c, 1 - c]$, that is, the gross valuation of the public good $\theta_i$ is uniformly distributed over $[0, 1]$. For any $c \in (0, 1)$, we can see that there is an interval of values of $p$ for which the possibility result is obtained.
As is clear from the graph (and it is straightforward to show analytically), for any value of \( p \in (0, 1) \), there exists an interval \( C_p \subset [0, 1] \) of the average cost \( c \) for which the possibility theorem holds. This implies that, when the society faces a series of public good provision problems with the average cost contained in the interval \( C_p \), then by carefully designing a democratic status quo, the society can implement the ex post efficient social choice function without violating the balanced-budget condition and the participation constraints.

This example suggests a qualitative answer to a question induced from our main theorem. Theorem 1 gives a necessary and sufficient condition for the possibility result. Then, one may wonder: what is the “best” majority rule which makes it most likely to satisfy the possibility condition. More precisely, how should the majority rule \( (m, p) \) change when we fix the shape of the type distribution and only move the mean of the distribution, keeping the possibility result intact? One may expect that a natural answer would be a majority rule with \( m = F(0)n \) in (2). For example, if \( F(0) = 0.2 \), then we may expect that the best rule would require at least 20% of the population to agree with the public good provision to best imitate the efficient provision. Indeed, the example above suggests that this intuition is incorrect. The best rule should be less extreme. For all \( c \), the possibility result is obtained only under majority with \( m = 1 \). Even for the values of \( c \) which are either smaller than one third or bigger than two thirds, the best majority rule should not be a veto rule, \( m = 0 \) or \( m = 2 \). The best rules require decreasing values of \( p \) while \( m \) is fixed as 1, suggesting that the best rules should become tighter as \( c = F(0) \) increases. In general, the best rule (if one exists) becomes tighter as \( F(0) \) increases, but not as much as the proportional rule; \( m \) increases, but \( m/n \) decreases, as \( F(0) \) increases.
4.2.2 Larger $n$

Suppose $n$ is larger than 2. Let us consider the symmetric case where $\tilde{\theta}_i \sim \text{unif}[−1/2, 1/2]$. Numerical examples show that the possibility result is obtained if and only if $n$ is even. As in the case of normal distribution, the probability of $U_{2n'+1}(\theta) > M_{2n'+1}(\theta)$ converges to one as $n' (\in \mathbb{N})$ goes to infinity.
5 Conclusion

This paper studies an institutional reform: the replacement of a voting mechanism for the provision of a public good by an efficient AGV mechanism. Voting mechanisms do not always efficiently decide on the provision of public goods. The present paper shows that there are cases in which all agents unanimously accept switching to the new ex post efficient mechanism. Therefore, one may be more optimistic about efficient institutional reforms when the status quo is a voting decision which involves coercion to contribute to the cost of collective goods.

The condition of the possibility theorem in Section 2 does not hold for all distributions of independent types. However, possibility results are robust for some symmetric distributions and increase in the population size.

Our paper is more optimistic about possible welfare gains than previous related studies. Even when ex post efficiency can not be achieved, the status quo may be relevant for the economy’s second best welfare level.

The insight of Cramton, Gibbons, and Klemperer’s (1987) paper is that any analysis of institutional reform under asymmetric information has to take the relevant status quo into account. The present paper applied this insight to the field of public economics. A useful extension of the present analysis is to study other more complex institutions such as representative democracies, voluntary provision schemes, or lobbying for collective provision.
A second interesting option for further research is to consider less complex reform proposals for collective decision procedures. AGV mechanisms are often criticized for being overly complicated and too unintuitive. This may indeed be an obstacle to their implementation in practice. It would be useful to find out whether more intuitive welfare enhancing mechanisms can also satisfy agents’ participation constraints.

6 Appendix

6.1 Proofs

We use the following lemmata in the proof of Theorem 2:

Lemma 2 \( \lim_{m \to \infty} 4^{-m} \sqrt{m} \binom{2m}{m} = 1/\sqrt{\pi} \).

Proof. Using Wallis’s product, we have

\[
\lim_{m \to \infty} \prod_{k=1}^{m} \left( \frac{2k}{2k-1} \right) \left( \frac{2k}{2k+1} \right) = \frac{\pi}{2}. \tag{6}
\]

Since

\[
\prod_{k=1}^{m} (2k - 1) = \frac{(2m - 1)!}{2^{m-1} (m - 1)!} \quad \text{and} \quad \prod_{k=1}^{m} (2k + 1) = \frac{(2m + 1)!}{2^{m} m!},
\]

(6) is equivalent to:

\[
\lim_{m \to \infty} \frac{4^{2m} (m!)^2 m! (m - 1)!}{(2m - 1)! (2m + 1)!} = \pi.
\]

Hence, we have

\[
\lim_{m \to \infty} \frac{4^{2m} (m!)^2}{m! (2m)!} = \pi \iff \lim_{m \to \infty} \frac{4^{m} (m!)^2}{\sqrt{m} (2m)!} = \sqrt{\pi},
\]

which is equivalent to what we claim. \( \blacksquare \)

For the examples of normal distribution, we mention directly to the most reluctant types \( \theta^+ \) and \( \theta^- \), since the set \( \Theta^+ \) and \( \Theta^- \) obviously consist of one element.

Lemma 3 Suppose \( \tilde{\theta}_i \sim N \left( 0, \sigma^2 \right) \). Then, a possibility theorem holds if and only if

\[
\max \{ |\theta^+|, |\theta^-| \} < \sigma \sqrt{\frac{n(n-1) \log \left( \frac{n}{n-1} \right)}{n}}.
\]
Proof. By definition,

\[ h(\tilde{\theta}_i) = \int_0^{\tilde{\theta}_i} s dF_i(s) \]

\[ = \frac{1}{\sqrt{2\pi (n-1)\sigma}} \int_0^{\tilde{\theta}_i} s \exp\left(-\frac{s^2}{2(n-1)\sigma^2}\right) ds \]

\[ = \frac{1}{\sqrt{2\pi (n-1)\sigma}} \left[ - (n-1)\sigma^2 \exp\left(-\frac{s^2}{2(n-1)\sigma^2}\right) \right]_0^{\tilde{\theta}_i} \]

\[ = \frac{\sqrt{n-1}\sigma}{\sqrt{2\pi}} \left( 1 - \exp\left(-\frac{\tilde{\theta}_i^2}{2(n-1)\sigma^2}\right) \right). \]

Hence,

\[ r = \int_{-\infty}^{\infty} h(\tilde{\theta}_i) dF_i(\tilde{\theta}_i) \]

\[ = \frac{\sqrt{n-1}\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left( 1 - \exp\left(-\frac{\tilde{\theta}_i^2}{2(n-1)\sigma^2}\right) \right) \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{\tilde{\theta}_i^2}{2\sigma^2}\right) d\tilde{\theta}_i \]

\[ = \frac{\sqrt{n-1}\sigma}{2\pi} \int_{-\infty}^{\infty} \left( 1 - \exp\left(-\frac{\tilde{\theta}_i^2}{2(n-1)\sigma^2}\right) \right) \exp\left(-\frac{\tilde{\theta}_i^2}{2\sigma^2}\right) d\tilde{\theta}_i \]

\[ = \frac{\sqrt{n-1}\sigma}{\sqrt{2\pi}} \left( 1 - \sqrt{\frac{n-1}{n}} \right). \]

In the last equality, we used:

\[ \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} \exp\left(-\frac{\tilde{\theta}_i^2}{2\sigma^2}\right) d\tilde{\theta}_i = 1 \]

and

\[ \int_{-\infty}^{\infty} \exp\left(-\frac{\tilde{\theta}_i^2}{2\sigma^2}\right) d\tilde{\theta}_i = \int_{-\infty}^{\infty} \exp\left(-\frac{\tilde{\theta}_i^2}{2\left(\sigma\sqrt{\frac{n-1}{n}}\right)^2}\right) d\tilde{\theta}_i = \left(\frac{n-1}{n}\right)^{\frac{1}{2}} \left(\frac{2\pi}{\sigma}\right). \]

Therefore, \( h(\theta^+) < r \) is equivalent to:

\[ \exp\left(-\frac{(\theta^+)^2}{2(n-1)\sigma^2}\right) > \sqrt{\frac{n-1}{n}} \Leftrightarrow |\theta^+| < a \sqrt{(n-1) \log\left(\frac{n}{n-1}\right)}. \]

\[ \Box \]

Proof of Theorem 2. By assumption, \( n \) is even and \( m = n/2 \). By symmetry, \( h(\theta^+) = h(\theta^-) \).

Hence, what we want to show is equivalent to \( h(\theta^+) < r \). Consider a sequence \((a_m)_{m=1}^{\infty}\) with:

\[ a_m = \frac{\sqrt{m}}{2m} \left(\frac{2m}{m}\right). \]
Since $a_{m+1}/a_m = (m + 1/2)/\sqrt{m(m + 1)} > 1$, $a_m$ is strictly increasing. By Lemma 2, we have
\[
\lim_{m \to \infty} a_m = 1/\sqrt{\pi}.
\]
Hence, for all $m \in \mathbb{N}$,
\[
\frac{\sqrt{m}}{4^m} \binom{2m}{m} < \frac{1}{\sqrt{\pi}}. \tag{7}
\]
Using Jensen’s inequality, we have
\[
\int_{2m-1}^{2m} \frac{1}{t} \, dt < \sqrt{\int_{2m-1}^{2m} \frac{1}{t} \, dt},
\]
which is equivalent to:
\[\begin{align*}
2 \left( \sqrt{2m} - \sqrt{2m-1} \right) &< \sqrt{\log \frac{2m}{2m-1}} \\
\iff \frac{1}{\sqrt{m}} &< \frac{1}{\sqrt{2}} \left( 1 + \sqrt{\frac{2m-1}{2m}} \right) \sqrt{\log \frac{2m}{2m-1}}. \tag{8}
\end{align*}\]
Since $\exp (-t^2/2)$ is concave for $|t| < 1$, $\forall x \in (0, 1)$,
\[\frac{x}{2} \left( 1 + \exp \left( -\frac{x^2}{2} \right) \right) < \int_0^x \exp \left( -\frac{t^2}{2} \right) \, dt.
\]
Applying this for $x = \sqrt{\log 2m/(2m-1)}$ ($m \geq 1$ implies $x < 1$),
\[\frac{1}{2} \sqrt{\log \frac{2m}{2m-1}} \left( 1 + \sqrt{\frac{2m-1}{2m}} \right) < \int_0^{\sqrt{\log \frac{2m}{2m-1}}} \exp \left( -\frac{t^2}{2} \right) \, dt. \tag{9}\]
Combining (7), (8) and (9), we have
\[\frac{1}{2} \cdot \frac{2m}{4^m} < \frac{1}{\sqrt{2\pi}} \int_0^{\sqrt{\log \frac{2m}{2m-1}}} \exp \left( -\frac{t^2}{2} \right) \, dt.
\]
Letting $n = m/2$,
\[\frac{1}{2^n} \binom{n}{n/2 - 1} < \Phi \left( \sqrt{\log \left( \frac{n}{n-1} \right)} \right) - \frac{1}{2}. \tag{10}\]
Now, remember that $\pi \left( \theta^+ \right) = \alpha^+$. By (5),
\[\Phi \left( \frac{\theta^+}{\sqrt{n-1} \sigma} \right) = \alpha^+.
\]
Therefore, by Lemma 3, $h \left( \theta^+ \right) < r$ if and only if
\[\theta^+ < \sigma \sqrt{(n-1) \log \left( \frac{n}{n-1} \right)} \Rightarrow \alpha^+ < \Phi \left( \sqrt{\log \left( \frac{n}{n-1} \right)} \right).
\]
By Lemma 1, this is equivalent to (10). \[\blacksquare\]
Proof of Theorem 3. Conditions (i)-(iii) are satisfied in the AGV mechanism. It remains to show (iv). Suppose $0 < a \leq b$. The proof for the case $a > b$ is analogous. By efficiency,

$$\pi(\tilde{\theta}_i) = \min \left\{ \frac{\tilde{\theta}_i + b}{a + b}, 1 \right\}.$$  

Then, by definition,

$$h(\tilde{\theta}_i) = \min \left\{ \frac{1}{2(a + b)} \tilde{\theta}_i^2, \frac{1}{2(a + b)} a^2 \right\}, \quad r = \frac{a^2}{2(a + b)^2} \left( b - \frac{1}{3} a \right).$$

Hence,

$$U(\tilde{\theta}_i) = \begin{cases} \frac{\tilde{\theta}_i}{a+b} \left( \frac{\tilde{\theta}_i}{a+b} + b \right) + \frac{a^2}{2(a+b)r} \left( b - \frac{1}{3} a \right) & \text{if } \tilde{\theta}_i < a, \\ \frac{\tilde{\theta}_i}{a+b} - \frac{2a^3}{3(a+b)^2} & \text{if } \tilde{\theta}_i > a. \end{cases}$$

Suppose $m = 1$ and $p \in (0,1)$. Then, $\alpha^+ = \frac{ap+b}{a+b}$, and $\alpha^- = \frac{bp}{a+b}$. Hence, $\theta^+ = ap$, and $\theta^- = \frac{-b(1-p)}{c}$. Therefore, (4) is equivalent to:

$$1 - \frac{a}{b} \sqrt{\frac{b-a/3}{a+b}} < p < \sqrt{\frac{b-a/3}{a+b}}.$$  

Note that $0 < a \leq b$ implies $0 < 1 - \frac{a}{b} \sqrt{\frac{b-a/3}{a+b}} < \sqrt{\frac{b-a/3}{a+b}} < 1$. When $a = c$ and $b = 1 - c$, it is straightforward to show that both the lower bound and the upper bound are strictly decreasing in $c$. □

Proof of Lemma 1. Define $y_i := \# \left\{ j \neq i \left| \tilde{\theta}_j > 0 \right. \right\}$. Then,

$$\alpha^+ = \Pr_{\tilde{\theta}_i} [y_i > m - 1] + p \Pr_{\tilde{\theta}_i} [y_i = m - 1],$$

$$\alpha^- = \Pr_{\tilde{\theta}_i} [y_i > m] + p \Pr_{\tilde{\theta}_i} [y_i = m].$$

Let $\lambda = F(0)$. Then,

$$\begin{align*}
\alpha^+ &= \sum_{i=m}^{n-1} \binom{n-1}{i} (1-\lambda)^i \lambda^{n-1-i} + p \sum_{i=m-1}^{n-1} \binom{n-1}{i} (1-\lambda)^i \lambda^{n-1-i}, \\
\alpha^- &= \sum_{i=m+1}^{n-1} \binom{n-1}{i} (1-\lambda)^i \lambda^{n-1-i} + p \sum_{i=m}^{n-1} \binom{n-1}{i} (1-\lambda)^i \lambda^{n-1-i}. 
\end{align*}$$

(11)

Since we assume that the net valuation is symmetrically distributed, $\lambda = 1/2$. Setting $m = n/2$ and $p = 1/2$, we obtain the result. □

6.2 Numerical examples

In this subsection, we describe how we obtained our numerical results.

6.2.1 Uniform distribution

Suppose that the net valuation is uniformly and symmetrically distributed: $\tilde{\theta}_i \sim \text{unif}[-1/2,1/2]$, and $n \geq 2$. Remember

$$U(\theta) = U(0) + \int_0^\theta \pi(s) \, ds.$$  

(12)
We first give an explicit formula of $\pi (\theta)$. Define

$$\varphi (n, k, \theta) := \binom{n}{k} \frac{(-1)^k}{n!} \left(\frac{n}{2} - k + \theta\right)^n.$$

**Proposition 1** Suppose that $\theta$ is identically, independently and uniformly distributed over the interval $[-\frac{1}{2}, \frac{1}{2}]$. Then,

$$\pi (\theta) = \sum_{k=0}^{B(n, \theta)} \varphi (n - 1, k, \theta)$$

where

$$B(n, \theta) = \begin{cases} \frac{n}{2} - 1 & \text{if } n \text{ is even} \\ \frac{n-3}{2} & \text{if } n \text{ is odd and } \theta \in \left[-\frac{1}{2}, 0\right] \\ \frac{n-1}{2} & \text{if } n \text{ is odd and } \theta \in \left[0, \frac{1}{2}\right] \end{cases}.$$

In order to prove Proposition 1, we use the following lemma.

**Lemma 4** Let $V_n(a)$ be the volume of the polyhedron:

$$\{(x_1, \cdots, x_n) | x_1 + \cdots + x_n \leq a, \ 0 \leq x_i \leq 1 (\forall i)\}$$

for $a \in (0, n)$. Then,

$$V_n(a) = \sum_{k=0}^{[a]} \binom{n}{k} (-1)^k \frac{(a-k)^n}{n!}$$

(13)

where $[a]$ is the largest integer which does not exceed $a$.

**Proof.** Proof is given by induction. Since $V_1(a) = a \in (0, 1)$, (13) holds for $n = 1$. Suppose $n \geq 2$ and (13) holds up to $n - 1$. Let $\tilde{a} = a - [a]$. Then by definition,

$$V_n(a) = \int_{0}^{\tilde{a}} V_{n-1}(a - a') \, da'$$

$$= \int_{0}^{\tilde{a}} \sum_{k=0}^{[a]} \binom{n-1}{k} (-1)^k \frac{(a-a'-k)^{n-1}}{(n-1)!} \, da' + \int_{\tilde{a}}^{1} \sum_{k=0}^{[a]-1} \binom{n-1}{k} (-1)^k \frac{(a-a'-k)^{n-1}}{(n-1)!} \, da'$$

$$= \sum_{k=0}^{[a]-1} \binom{n-1}{k} \frac{(-1)^k}{(n-1)!} \int_{0}^{\tilde{a}} (a-a'-k)^{n-1} \, da' + \binom{n-1}{[a]} \frac{(-1)^{[a]}}{(n-1)!} \int_{0}^{\tilde{a}-[a]} (a-a'-[a])^{n-1} \, da'.$$

Taking the integral,

$$V_n(a) = \sum_{k=0}^{[a]-1} \binom{n-1}{k} \frac{(-1)^{k+1}}{n!} [(a-1-k)^n - (a-k)^n] + \binom{n-1}{[a]} \frac{(-1)^{[a]}}{n!} (a-[a])^n$$

$$= \sum_{k=0}^{[a]-1} \binom{n-1}{k} \frac{(-1)^{k+1}}{n!} (a-(k+1))^n + \sum_{k=0}^{[a]-1} \binom{n-1}{k} \frac{(-1)^k}{n!} (a-k)^n$$

$$+ \binom{n-1}{[a]} \frac{(-1)^{[a]}}{n!} (a-[a])^n.$$
Changing the label of $k$ in the first term, we obtain:

$$V_n(a) = \sum_{k=0}^{\lfloor a \rfloor} \binom{n-1}{k} \frac{(-1)^k}{n!} (a-k)^n + \sum_{k=0}^{\lfloor a \rfloor} \binom{n-1}{k} \frac{(-1)^k}{n!} (a-k)^n = \sum_{k=0}^{\lfloor a \rfloor} \binom{n}{k} \frac{(-1)^k}{n!} (a-k)^n.$$ 

\[ \Box \]

**Proof of Proposition 1.** Let $\theta_i' = \frac{1}{2} - \tilde{\theta}_i$ and $s(\tilde{\theta}_i) = \frac{n-1}{2} + \tilde{\theta}_i$. Then,

$$\pi(\tilde{\theta}_i) = \Pr_{\tilde{\theta}_i} \left[ \tilde{\theta}_i + \sum_{j \neq i} \tilde{\theta}_j \geq 0 \right] = \Pr_{\tilde{\theta}_i} \left[ \sum_{j \neq i} \tilde{\theta}_j \leq s(\tilde{\theta}_i) \right].$$

Since $\theta_j' \sim \text{unif}[0,1]$, $\pi(\tilde{\theta}_i) = V_{n-1} \left( s(\tilde{\theta}_i) \right)$. (i) If $n$ is even, $s(\tilde{\theta}_i) = \frac{n}{2} - 1$. (ii) If $n$ is odd and $\tilde{\theta}_i > 0$, then $s(\tilde{\theta}_i) = \frac{n-1}{2}$. (iii) If $n$ is odd and $\tilde{\theta}_i < 0$, then $s(\tilde{\theta}_i) = \frac{n-3}{2}$. By Lemma 4, Proposition 1 follows. \[ \Box \]

Now, we give a formula for $U(0)$.

**Proposition 2** Suppose that $\theta$ is identically, independently and uniformly distributed over the interval $[-\frac{1}{2}, \frac{1}{2}]$. Then,

$$U(0) = \int_0^{1/2} \pi(\theta) (4\theta - 1) d\theta.$$ 

To give a proof, we use the following lemma.

**Lemma 5** $\pi(\theta) + \pi(-\theta) = 1$.

**Proof.** Let $P(x)$ be a polynomial of degree $n$, and let $a_n$ be the coefficient of $n$-th degree. Then, in general, we have

$$\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{n!} P(n-k) = a_n.$$ 

Applying this to $\varphi(n,k,\theta)$, we obtain

$$\sum_{k=0}^{n-1} \varphi(n,k,\theta) = 1. \quad (14)$$

Moreover,

$$\varphi(n,k,-\theta) = \binom{n}{k} \frac{(-1)^k}{n!} \left( \frac{n}{2} - k - \theta \right)^n = \binom{n}{n-k} \frac{(-1)^{n-k}}{n!} \left( \frac{n}{2} + k + \theta \right)^n$$

$$= \varphi(n,n-k,\theta). \quad (15)$$

(i) Suppose that $n$ is even. Then, by (14) and (15),

$$\pi(\theta) + \pi(-\theta) = \sum_{k=0}^{\frac{n}{2}-1} \varphi(n-1,k,\theta) + \sum_{k=0}^{\frac{n}{2}-1} \varphi(n-1,n-1-k,\theta)$$

$$= \sum_{k=0}^{\frac{n}{2}-1} \varphi(n-1,k,\theta) + \sum_{k=\frac{n}{2}}^{n-1} \varphi(n-1,k,\theta) = 1.$$
(ii) Suppose that $n$ is odd and $\theta \geq 0$. Then, by (14) and (15),

$$
\pi (\theta) + \pi (-\theta) = \sum_{k=0}^{n-1} \varphi (n-1, k, \theta) + \sum_{k=0}^{n-3} \varphi (n-1, n-1-k, \theta)
$$

$$
= \sum_{k=0}^{n-1} \varphi (n-1, k, \theta) + \sum_{k=\frac{n-1}{2}}^{n-1} \varphi (n-1, n-1-k, \theta) = 1.
$$

Now, we prove Proposition 2.

**Proof of Proposition 2.** By definition, $U (\theta) - \theta \pi (\theta) = r - h (\theta)$ and $r = \mathbb{E}_\theta [h (\theta)]$. Hence, $
\mathbb{E}_\theta [U (\theta) - \theta \pi (\theta)] = 0$. Using (12),

$$
U (0) = -\mathbb{E}_\theta \left[ \int_0^\theta \pi (s) ds - \theta \pi (\theta) \right].
$$

In our uniform case,

$$
U (0) = -\int_{-1/2}^{1/2} \left( \int_0^\theta \pi (s) ds - \theta \pi (\theta) \right) d\theta.
$$

Using Lemma 5,

$$
\int_0^\theta \pi (s) ds - \theta \pi (\theta) = \int_0^{-\theta} \pi (s) ds + \theta \pi (-\theta).
$$

Hence,

$$
U (0) = -2 \int_0^{1/2} \left( \int_0^\theta \pi (s) ds - \theta \pi (\theta) \right) d\theta
$$

$$
= -2 \left( \int_0^{1/2} \int_0^\theta \pi (s) ds d\theta - \int_0^{1/2} \theta \pi (\theta) d\theta \right)
$$

$$
= -2 \left( \int_0^{1/2} \int_s^{1/2} \pi (s) d\theta ds - \int_0^{1/2} \theta \pi (\theta) d\theta \right)
$$

$$
= -2 \left( \int_0^{1/2} \pi (s) \left( \frac{1}{2} - s \right) ds - \int_0^{1/2} \theta \pi (\theta) d\theta \right)
$$

$$
= \int_0^{1/2} \pi (\theta) (4\theta - 1) d\theta.
$$

Combining Proposition 1 and 2, we can compute $U (\theta)$. As we mentioned in the main text, a possibility result is obtained if and only if $n$ is even.
6.2.2 A modified majority rule

When \( n \) is odd, a possibility result is obtained by slightly modifying the status quo majority rule. For each \( p \in \left[ \frac{1}{2}, 1 \right] \), we define a modified majority rule as follows, instead of (2),

\[
g_p (\xi) = \begin{cases} 
1 & \text{if } \sum \xi_i > m + 1 \\
p & \text{if } \sum \xi_i = m + 1 \\
1 - p & \text{if } \sum \xi_i = m \\
0 & \text{if } \sum \xi_i < m 
\end{cases}
\]

Then, the interim probability that the public good is provided is

\[
M_p (\tilde{\theta}_i) = \begin{cases} 
(1-p)^{n-1} \left( \begin{array}{c} m-1 \\ n-m \end{array} \right) + p^{n-1} + \sum_{k=m+1}^{m+1} \left( \begin{array}{c} n-1 \\ k \end{array} \right) \tilde{\theta}_i & \text{if } \tilde{\theta}_i \geq 0 \\
(1-p)^{n-1} \left( \begin{array}{c} m-1 \\ n-m \end{array} \right) + p^{n-1} + \sum_{k=m+2}^{m+2} \left( \begin{array}{c} n-1 \\ k \end{array} \right) \tilde{\theta}_i & \text{if } \tilde{\theta}_i \leq 0
\end{cases}
\]

Numerical examples show that a possibility result is obtained when \( p \) is sufficiently close to 1/2.

References


